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TORSIONAL RIGIDITY OF CANTILEVER WINGS WITH CONSTANT SPAR AND RIB SECTIONS

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TORSIONAL RIGIDITY OF CANTILEVER WINGS WITH CONSTANT SPAR AND RIB SECTIONS.*

By Giuseppe Gabrielli.

During recent years several very thorough and interesting treatises have been devoted to the investigation of systems of spars and ribs. These researches represent considerable progress, especially for the statics of cantilever wings. From them we have obtained a clearer idea of the effect of the ribs and have also been enabled, in a few special cases, to calculate, by relatively simple methods, the magnitude of the principal effect of the union of the spars and ribs, namely, the reduction of the bending moment at the fixed ends of the spars.

The present paper treats less of the effect of the union than of its influence on the torsional rigidity of the wing. The calculations are carried out for a two-spar wing of constant cross section, in which the ribs are replaced by a continuous member of constant rigidity (assumption of an infinite number of ribs). The bending rigidity of the ribs is assumed to be very great in comparison with their torsional rigidity, as is the case in actual practice.

*"Ueber die Torsionssteifigkeit eines freitragenden Flügels mit konstantem Holm- und Rippenquerschnitt" from Luftfahrtforschung, June 26, 1928, pp. 79-90.

The case under consideration is strongly idealized. In reality the bending and torsional rigidities of the spars vary throughout the length of the wing. Nevertheless, the accurate numerical evaluation of this idealized case is of some interest, because it gives us an accurate idea of how the ratio of the bending to the torsional rigidity of the spars affects the torsional rigidity of the wing as a whole. Above all, it enables us to tell when the torsion of the whole wing may be regarded merely as the result of the various flexures of the spars and when, on the other hand, the wing may be regarded simply as a twisted girder. In this sense the designer can make approximate calculations, even in the most complicated cases, on the basis of the following method.

Before proceeding to the calculation, we will take a look at the reports which have already appeared, especially during the preparation of the present paper.

1. We find an article by Ballenstedt.* He discusses a wing structure with two like spars of constant cross section, and calculates, by the usual methods for statically indeterminate systems, the effect of the ribs in the case of unsymmetrical loading of the spars. The wing considered had nine like ribs. It was assumed that the middle rib was fixed, so that the calculation simply concerns a wing structure consisting of

*"Der Einfluss der Spante auf die Festigkeit der Holme." Technische Berichte der Flugzeugmeisterei, Vol. III, (1917-1918), pp. 100-111.

two spars and four ribs. On the assumption that the same load was applied at every joint, Ballenstedt calculates the course of the bending and torsional moments along the spars and shows the diminution of the maximum bending moment due to the union of the ribs.

2. Dr. Thalau's papers contain much valuable material.* He also discusses a rigidly supported cantilever wing, consisting of two spars and ribs. He defines this structure as a lattice girder. He chooses, as the expression of the reciprocal effect of the union of the spars, the diminution of the bending moment at the point of fixation of the spars (fixation cross section) and shows that the end rib has the greatest effect and that the other ribs have only a negligible effect on the bending moment at the point of fixation. Hence he investigates especially the shearing forces produced by the end rib in the case of like tapering spars..

3. Biezeno, Koch and Koning replace the rib union by a continuous union effect, as is done in the present paper.** They disregard the torsional rigidity of the ribs and consider them as inflexible. They give an approximation method for the case when the rigidities of the two spars have the same ratio

*"Zur Berechnung freitragender Flugzeugflügel." Z.F.M., 1924, p.103; 1925, pp.86-87. "Ueber die Verbundwirkung von Rippen im freitragenden, zweiholmigen und verspanungslosen Flugzeugflügel. Z.F.M., 1925, No. 20.
**"Ueber die Berechnung von freitragenden Flugzeugflügeln." Zeitschrift für Mech. und angew. Math. 1926, pp. 97-105.

in every cross section. Lastly, they investigate a wing with constant spar and rib cross sections on the assumption of uniform loading and calculate the diminution in the bending moment due to the union.

4. Reissner investigates the case of a wing situated above the fuselage and having two like spars, on the assumption that the stresses are constant in every cross section of the spars (wings of constant strength).* Reissner obtains exceedingly simple formulas on the distribution of the bending moment and on the reciprocal reduction of the load on the spars, for both symmetrical and unsymmetrical loading. For this case we can obtain a simple expression of the magnitude, which may be defined as the torsional rigidity of the whole wing. Reissner then employs this principle to determine the limits of static stability of wings exposed to the action of the air.

5. Von Karman, in the winter semester of 1925/6 in his lectures on "Introduction to the Study of Airplanes," gave the general equations likewise on the assumption of an infinite number of ribs, but without the hypothesis of inflexible ribs and with consideration of the torsional rigidity of the ribs for a two-spar system and showed the influence lines for the loading of a spar by a single load applied to the other spar. We shall follow Von Karman's lectures in the application of the differential equations.

*"Neuere Probleme aus der Flugzeugstatik. Verbundwirkung des Flügelkörpers." Z.F.M., 1926, pp. 181-185.

Bending and Torsion of a Cantilever Wing

We investigated an unbraced cantilever wing structure, formed by two spars of uniform cross section and a number of equal ribs distributed at regular intervals along the spars. The structure was rigidly supported at one end. For the following calculation, however, we assumed an infinite number of elementary ribs regularly distributed along the spars and replacing, in their effect, the ribs actually present. The ribs were assumed to be inflexible, but to have finite torsional rigidity with respect to their longitudinal axes.

If both spars were stressed by separate or distributed loads proportional to the bending rigidities of the corresponding spars, the latter would be equally deflected. In this case the behavior of the spars was not affected by the ribs.

Let $p_1 (x)$ and $p_2 (x)$ denote the loads distributed along the spars; A_1 and A_2, their corresponding bending rigidities; and x, the coordinates along the spars. If we then have the relation

$$\frac{p_1 (x)}{A_1} = \frac{p_2 (x)}{A_2} \tag{1}$$

the wing is simply bent. By a corresponding arrangement of the spars with respect to the wing chord, the above condition can be fulfilled in horizontal flight. This condition is not fulfilled in diving and gliding flight, due to the travel of the center of pressure, and both spars are twisted about their longitudinal axis due to their union through the ribs. In other words, they are stressed simultaneously with respect to both bending and torsion. From the assumption of absolute inflexibility of the ribs, it follows directly that the angles of torsion of both spars have the same value in every wing section. We call this angle the angle of torsion of the wing in the corresponding cross sections.

If the resultant of the external forces for all the wing sections lies between the two spars and divides the distance b (Fig. 1) into two parts b_1 and b_2, so that $b_1/b_2 = A_2/A_1$, the wing then undergoes a simple bending. The point O is called the "elastic center" of the given wing section. The line connecting the elastic centers of all the wing sections is called the "elastic axis" of the wing. Obviously, the location of the elastic axis of a wing depends on the location of the spars with respect to the wing chord and on the bending rigidity of the spars. The elastic axis is a straight line in wings with constant or linearly diminishing bending rigidity.

If we consider a wing with spars of constant rigidity and subjected to a load p distributed uniformly throughout their length, the wing is then subjected to a uniformly distributed load of $2\,p$ and a moment of $p\,(b_1 - b_2)$. If we assume, on the other hand, that the spar cross sections, belonging to the same wing section, are subjected to two equal and opposite forces, the elastic axis will then suffer no deflection. This is designated as the case of simple torsion. It is obvious that the general case can then be limited, by a corresponding shifting of the load, to these two individual cases. Since the calculation of the simple bending offers no difficulty, we can limit our calculations to the case of simple torsion.

The Torsional Rigidity of the Wing Section and of the Wing Itself

While, for an elastic girder (generally of variable cross section) the ratio between the torsion per unit length and the torsional moment in each cross section has a value dependent only on the cross section itself and characterizing the torsional rigidity of the section, this is not the case for a wing with spars of constant cross section.

In fact, the torsional angle ϑ of a wing does not depend alone on the torsional moments in the two-spar sections, but also on the flexibility of the spars. The torsional moment is, in fact, equilibrated in every cross section in part by the torsional moments of the spars and also by the moment of the shearing forces in the spar sections. Obviously, these depend on the flexibility of the spars. Since the flexibility of the spars depends on the load distribution, it follows that the value $\vartheta' = d\vartheta/dx$ (x being the coordinate along the wing axis) in each wing section depends not only on the magnitude of the torsional moment, but also on the load distribution. In other words, the value ϑ' for each section of a cylindrical girder is proportional to the prevailing torsional moment, independently of the value of the moment M_t in the other sections of the girder. For a wing section, however, ϑ' depends on the value and on the distribution of the torsional moment along the wing.

For every wing section we can therefore define by the equation

$$M_t = - B\,\vartheta' \tag{2}$$

a torsional rigidity B, which, however, depends both on the characteristic quantities (bending and torsional rigidity) for the spars and ribs and also on the distribution of the torsional moment along the wing. Furthermore, by the expression

$$B_f = \frac{M_t\,l}{\vartheta_0} \tag{3}$$

we can introduce a torsional rigidity B_f for the whole wing. Thereby M_t is the mean torsional moment of the wing; l, its length; and ϑ_0 its distortion.

We will undertake to determine the values B and B_f for a wing of constant spar and rib section in the following cases.

1. Wing with a torsional moment increasing linearly from the tip toward the root.

2. Wing with a torsional moment applied at the tip.

In what follows it will be shown for both cases, to what extent the torsional moment in each wing section is balanced by the torsion of the spars and by the shearing forces in the spar section.

The Differential Equations of the Spar System on the Assumption of Inflexible Ribs

We place the x-axis of the coordinate parallel to the spars with the origin at the wing tip, the direction from the wing tip toward the wing root being considered positive. The downward direction of the axes ζ_1 and ζ_2 is considered positive (Fig. 2). The following symbols are used.

B_r, the torsional rigidity of a rib per unit length;

$n = 1/\alpha$, the number of ribs per centimeter length of the spars;

β, the torsional rigidity of the ribs per centimeter length of the spars ($\beta = n\ B_r$);

ζ_1 and ζ_2, the bending deflections of the front and rear spars, considered positive downward ($\zeta = \zeta_1 - \zeta_2$);

ϑ, the torsional angle of any wing section with the abscissa x (i.e., the torsional angle of the spars with respect to the fixed root section, read positively from the wing tip in the direction traveled by the hands of a clock);

A_1 and A_2, the respective torsional rigidities of the front and rear spars;

b, the constant distance between the spars;

μ_1 and μ_2, the torsional moments per centimeter length, which the ribs transmit to the spars, these moments being considered positive in the direction of the motion of the hands of a clock as seen from the wing tip;

τ_1 and τ_2, the bending moments per centimeter length, which the ribs transmit to the spars. The same numbers with opposite signs represent the torsional moments at the ends of the ribs. τ_1 and τ_2 (applied to the spars) are considered positive, when they act in the direction which conveys the positive ζ axis in the positive direction of the x axis (Fig. 2);

p_1 and p_2, the external loading of the spars per centimeter length;

q_1 and q_2, the loads per centimeter length, which the ribs exert on the spars due to the distortion of the wing.

a) The equilibrium of the ribs.- The condition of equilibrium for the forces (Fig. 3) reads $q_1 + q_2 = 0$ and for the moments

$$q_1 = \frac{\mu_1 + \mu_2}{b} \tag{4}$$

From these two equations we obtain

$$q_2 = -\frac{\mu_1 + \mu_2}{b} \tag{5}$$

The condition of equilibrium for an elementary rib, with respect to rotation about its own axis, is

$$\tau_1 + \tau_2 = 0,$$

that is,

$$\tau_1 = -\tau_2 = \tau \tag{6}$$

b) The equations for the bending of the spars.- for any cross section (abscissa x) of the front spar, the fundamental equation is

$$A_1 \frac{d^2 \zeta_1}{d\, x^2} = M_1,$$

in which M_1 denotes the moment of the forces acting, between the wing tip and the given cross section, on the center of gravity of said cross section.

If we designate by Q_1 the shearing force in the given cross section and consider it positive when directed upward (Fig. 2), we have

$$\frac{d\ M_1}{d\ x} = -\ Q_1 + \bar{\tau}$$

and consequently,

$$A_1 \frac{d^3\ \zeta_1}{d\ x^3} = -\ Q_1 + \bar{\tau} \tag{7}$$

Moreover, (Fig. 2)

$$\frac{d\ Q_1}{d\ x} = -\ (p_1 + \bar{q}_1)$$

and hence

$$A_1 \frac{d^4\ \zeta_1}{d\ x^4} = \bar{p}_1 + \bar{q}_1 + \frac{d\tau}{dx} \tag{8}$$

For the rear spar, with the aid of equation (6), we likewise obtain

$$A_2 \frac{d^3\ \zeta_2}{d\ x^3} = -\ Q_2 - \tau \tag{7a}$$

and

$$A_2 \frac{d^4\ \zeta_2}{d\ x^4} = p_2 + q_2 - \frac{d\tau}{dx} \tag{8a}$$

c) The torsional equations for the spars.- For the front spar the torsional equation reads $B_1\ \vartheta' = -\ M_1$, in which M_1 denotes the torsional moment (about the center of gravity of the given cross section) of all the forces acting on the spar between this section and the wing tip. Taking into account the hypotheses concerning the positive direction of ϑ, we must consider M_1 positive when, as seen from the wing tip, it acts in the direction traveled by the hands of a clock. It is $dM_1/dx = \mu_1$, in which μ_1 is the torsional load of the front spar (and also the bending load of the ribs), considered positive when acting in the direction of motion of the hands of a clock, as seen from the wing tip. We obtain

$$B_1 \; \vartheta'' = - \frac{d\,M_1}{d\,x} = - \mu_1;$$

and, for the rear spar, $B_2 \; \vartheta'' = - \mu_2$. The addition of these two equations gives

$$(B_1 + B_2)\,\vartheta'' = - (\mu_1 + \mu_2).$$

With the aid of this equation we can transform equations (4) and (5) into

$$q_1 = - \frac{B_1 + B_2}{b}\,\vartheta'' \tag{4a}$$

and

$$q_2 = \frac{B_1 + B_2}{b}\,\vartheta'' \tag{5a}$$

d) Relation between the Flexure ϑ and the Torsional Angle ϑ of the Spars

From Figure 3, it follows that

$$\vartheta = - \frac{\zeta_1 - \zeta_2}{b} = - \frac{\zeta}{b} \tag{9}$$

e) Relation between the Flexure of the Spars and the Torsion of the Ribs

Bearing in mind (Fig. 2) that the angle of torsion of a rib of the length b is given by

$$\frac{d\zeta}{dx} = \frac{d\zeta_1}{dx} - \frac{d\zeta_2}{dx}$$

at every point of the abscissa x (cf. equation (9)), we obtaim

$$\tau = \frac{\beta}{b} \frac{d\zeta}{dx} \tag{10}$$

and, from this,

$$\frac{d\tau}{dx} = \frac{\beta}{b} \frac{d^2\zeta}{dx^2} \tag{11}$$

If we introduce the value of $d\tau/dx$ into equation (8) and (8a), we obtain

$$A_1 \frac{d^4\zeta_1}{dx^4} = p_1 + \bar{q}_1 + \frac{\beta}{b} \frac{d^2\zeta}{dx^2},$$

$$A_2 \frac{d^4\zeta_2}{dx^4} = p_2 + \bar{q}_2 - \frac{\beta}{b} \frac{d^2\zeta}{dx^2}.$$

On combining these two equations, we obtain

$$\frac{d^4\zeta_1}{dx^4} - \frac{d^4\zeta_2}{dx^4} = \frac{1}{A_1} p_1 - \frac{1}{A_2} p_2 + \frac{1}{A_1} \bar{q}_1 -$$

$$\frac{1}{A_2} \bar{q}_2 + \frac{\beta}{b} \frac{d^2\zeta}{dx^2} \left(\frac{1}{A_1} + \frac{1}{A_2}\right).$$

If we put $\frac{1}{A} = \frac{1}{A_1} + \frac{1}{A_2}$ and consider equations (4a), (5a), and (9), we obtain

$$\vartheta'''' - \frac{B_1 + B_2 + \beta b}{A b^2} \vartheta'' + \frac{1}{b}\left(\frac{p_1}{A_1} - \frac{p_2}{A_2}\right) = 0.$$

For simplification we put

$$\lambda^2 = \frac{B_1 + B_2}{A b^2}$$

and

$$K^2 = 1 + \frac{\beta b}{B_1 + B_2} = 1 + \frac{b}{d} \frac{B r}{B_1 + B_2}$$

and obtain

$$\vartheta'''' - K^2 \lambda^2 \vartheta'' + \frac{1}{b}\left(\frac{p_1}{A_1} - \frac{p_2}{A_2}\right) = 0, \qquad (12)$$

as the final result.

Relation between the Shearing Forces and Torsional Moments in the Cross Sections of the Wing Spars Subjected to Simple Torsion

The torsional moment is held in equilibrium in every wing section by the shearing forces and torsional moments prevailing in the section. We will consider a wing subjected to simple torsion, M_t being the torsional moment in any wing section. The condition of equilibrium for the portion of the wing included between the given section and the wing tip (the forces and moments in this portion acting in the opposite direction to that indicated in Figure 4) is $Q_1 = - Q_2 = Q$ for the forces and

$$-Qb - M_1 - M_2 + Mt = 0$$

for the moments. Since

$$M_1 = - B_1 \vartheta' \quad \text{and} \quad M_2 = - B_2 \vartheta'$$

it follows that

$$-Qb + (B_1 + B_2)\,\vartheta' + Mt = 0. \qquad (13)$$

It now follows from equations (7) and (7a) that

$$Q_1 = - A_1 \frac{d^3 \zeta_1}{d\,x^3} + \frac{\beta}{b}\frac{d\zeta}{dx} \qquad Q_2 = A_2 \frac{d^3 \zeta_2}{d\,x^3} + \frac{\beta}{b}\frac{d\zeta}{dx}$$

and therefrom

$$Q = - \frac{A_1\,\zeta_1''' - A_2\,\zeta_2'''}{2} - \frac{\beta}{b}\,\zeta' \qquad (14)$$

$$A_1\,\zeta_1''' + A_2\,\zeta_2''' = 0. \qquad (15)$$

According to system of notation employed

$$A_1\,\zeta''' = A_1\,(\zeta_1''' - \zeta_2''') \qquad (16)$$

$$A_2\,\zeta''' = A_2\,(\zeta_1''' - \zeta_2''') \qquad (17)$$

From (15) and (16) we obtain by subtraction

$$- A_1\,\zeta''' = (A_1 + A_2)\,\zeta_2''' .$$

From (15) and (17) we obtain by addition

$$A_2\,\zeta''' = (A_1 + A_2)\,\zeta_2''' ,$$

from which it follows that

$$\zeta_1''' = \frac{A_2}{A_1 + A_2}\,\zeta''' \qquad \zeta_2''' = - \frac{A_1}{A_1 + A_2}\,\zeta'''.$$

Multiplying by A_1 and A_2 we obtain

$$A_1\ \zeta_1''' = A\ \zeta''' \qquad A_2\ \zeta_2''' = -\ A\ \zeta''',$$

whence

$$A\,\zeta''' = \frac{A_1\ \zeta_1''' - A_2\ \zeta_2'''}{2}\ .$$

Equation (14) then becomes

$$\bar{Q} = -\ A\,\zeta''' + \frac{\bar{\beta}}{b}\ \zeta'$$

and, with the use of equation (9)

$$\bar{Q} = A\ b\ \bar{\vartheta}''' - \beta\ \vartheta'$$

and the condition of equilibrium (13) becomes

$$\vartheta''' - \frac{1}{Ab^2}\ (\beta b + \bar{B}_1 + B_2)\ \vartheta' = \frac{M_t}{Ab^2}$$

or, with the use of the chosen abbreviation,

$$\vartheta''' - K^2\lambda^2\ \vartheta' = \frac{M_t}{Ab^2}\ . \tag{18}$$

From equation (13) we obtain

$$\bar{Q} = \frac{B_1 + B_2}{b}\ \vartheta' \bar{+} \frac{M_t}{b} \tag{19}$$

This equation shows the relation between the shearing forces and torsional moments of the spars in every wing section.

Case I. Spars with Uniformly Distributed Load
(p_1 and p_2 being constants)

The fundamental equation (12) holds good for any distribution of the loads p_1 and p_2 along the spars, hence also for the case when a single force acts on the wing tip. We will consider the case when the spar loading is uniformly distributed, i.e., when p_1 and p_2 are constants. The member $\frac{1}{b}\left(\frac{p_1}{A_1} - \frac{p_2}{A_2}\right)$ of the differential equation is then a constant. The general equation for this case is

$$\vartheta = C_1\ e^{K\lambda x} + C_2\ e^{-K\lambda x} + C_3\ x + C_4 + \frac{1}{2b\ K^2\ \lambda^2}\left(\frac{p_1}{A_1} - \frac{p_2}{A_2}\right) x\ ,$$

in which the four integration constants C_1, C_2, C_3, and C_4 are to be determined from the following limiting conditions. For $x = l$ we have, as a result of the rigid fixation, $\vartheta = 0$ and $\vartheta' = 0$, instead of $x = 0$, that is, at the wing tip we must have $\vartheta'' = 0$, due to the vanishing bending moment ζ''. If it is noted that the torsional moment is zero at the wing tip, it follows for equation (18) that

$$\vartheta'''_{x=o} - K^2\ \lambda^2\ \vartheta'_{x=o} = 0$$

is the fourth limiting condition.

For these limiting conditions the constants become

$$C_1 = -\frac{1}{2b\ K^2\ \lambda^2}\left(\frac{p_1}{A_1} - \frac{p_2}{A_2}\right)\frac{K\ \lambda\ l + e^{-K\lambda l}}{K^2\ \lambda^2\ \cosh K\ \lambda\ l}$$

$$C_2 = \frac{1}{2b\ K^2\ \lambda^2}\left(\frac{p_1}{A_1} - \frac{p_2}{A_2}\right)\frac{K\ \lambda\ l - e^{K\lambda l}}{K^2\ \lambda^2\ \cosh K\ \lambda\ l}$$

$$C_3 = 0$$

$$C_4 = \frac{1}{2b\ K^2\ \lambda^2}\left(\frac{p_1}{A_1} - \frac{p_2}{A_2}\right)\frac{1}{K^2\ \lambda^2\ \cosh K\ \lambda\ l}\left\{K^2\lambda^2\ l^2\ \cosh K\lambda l - 2\ K\ \lambda\ l\ \sinh K\ \lambda\ l - 2\right\},$$

and we have

$$\vartheta = \frac{\dfrac{p_1}{A_1} - \dfrac{p_2}{A_2}}{2b\ K^4\ \lambda^4\ \cosh K\ \lambda\ l}\left\{2\ \cosh K\ \lambda\ (l - x) - 2\ K\ \lambda\ l\ (\sinh K\lambda l - \sinh K\ \lambda\ x) + K^2\ \lambda^2\ (l^2 - x^2)\ \cosh K\ \lambda\ l - 2\right\}.$$

When $\dfrac{p_1}{A_1} = \dfrac{p_2}{A_2}$, then $\vartheta = 0$.

Hence we can always resolve the case when p_1 and p_2 have any desired values into two cases, that of simple bending, in which the load is proportional to the bending rigidity, and that of simple torsion, in which the loads of the two spars are alike and opposite.

For the latter case, we obtain the formula

$$p = A\left(\frac{p_1}{A_1} - \frac{p_2}{A_2}\right).$$

From here on we will confine our attention to this case. The load can be so expressed that the wing will be subjected to a linearly variable moment of the magnitude $M_t = p\ b\ x$.

a) The torsional angle is expressed by the equation

$$\vartheta = \frac{p}{2b\ A\ K^4\ \lambda^4\ \cosh K\lambda l}\Big\{2 \cosh K\lambda\ (l - x) - 2\ K\ \lambda\ l(\sinh K\ \lambda\ l - \sinh K\ \lambda\ x) + K^2\ \lambda^2\ (l^2 - x^2) \cosh K\ \lambda\ l - 2\Big\}.$$

Since

$$b^2\ A\ \lambda^2 = B_1 + B_2,$$

we can write:

$$\vartheta = \frac{p\ b\ l^2}{2\ K^2\ (B_1 + B_2)}\ \frac{1}{K^2\ \lambda^2\ l^2 \cosh K\ \lambda\ l}\Big\{2 \cosh K\lambda\ (l - x) - 2\ K\ \lambda\ l\ (\sinh K\ \lambda\ l - \sinh K\ \lambda\ x) + K^2\ \lambda^2\ (l^2 - x^2) \cosh K\ \lambda\ l - 2\Big\} \quad (20)$$

If we put $B_i = K^2\ (B_1 + B_2)$, in which B_i may be regarded as the sum of the torsional rigidities of the spars and of the ribs situated within the distance b, then

$$\vartheta = \frac{p\ b\ l^2}{2\ B_i}\ \frac{1}{K^2\ \lambda^2\ l^2 \cosh K\ \lambda\ l}\Big\{2 \cosh K\ \lambda\ (l - x) - 2\ K\ \lambda\ l\ (\sinh K\ \lambda\ l - \sinh K\ \lambda\ x) + K^2\ \lambda^2\ (l^2 - x^2) \cosh K\ \lambda\ l - 2\Big\}.$$

The term $p\ b\ l^2/2\ B_i$ represents the angle of torsion ϑ_{io} of a girder of the torsional rigidity B_i. We then obtain

$$\frac{\vartheta}{\vartheta_{io}} = \frac{1}{(K \lambda l)^2 \cosh K \lambda l} \left\{ 2 \cosh K \lambda (l - x) - 2 K \lambda l (\sinh K \lambda l - \sinh K \lambda x) + K^2 \lambda^2 (l^2 - x^2) \cosh K \lambda l - 2 \right\} \quad (21)$$

ϑ/ϑ_{io} is plotted against x/l for different values of $K \lambda l$ in Figure 5 (Table I).

TABLE I (Fig. 5)

Values of:

$$\frac{\vartheta}{\vartheta_{io}} = \frac{1}{(K\lambda l)^2 \cosh K \lambda l} \left\{ 2 \cosh K \lambda (l - x) - 2 K\lambda l (\sinh K\lambda l - \sinh K\lambda x) + K^2 \lambda^2 (l^2 - x^2) \cosh K \lambda l - 2 \right\}$$

x/l	$K\lambda l=1$	$K\lambda l=2$	$K\lambda l=3$	$K\lambda l=4$
0.0	0.181	0.403	0.530	0.630
0.2	0.134	0.314	0.440	0.537
0.4	0.888	0.217	0.325	0.404
0.6	0.046	0.124	0.191	0.247
0.8	0.014	0.038	0.057	0.087
1.0	0.000	0.000	0.000	0.000

From equation (21) we obtain for $x = 0$

$$\frac{\vartheta_o}{\vartheta_{io}} = 1 + \frac{2}{(K\lambda l)^2} - \frac{2 \tanh K\lambda l}{K \lambda l} - \frac{2}{(K\lambda l)^2 \cosh K\lambda l} .$$

As $K \lambda l$ approaches ∞, this ratio approaches 1, which means that the wing undergoes a torsion equal to that of a girder of the same length with the torsional rigidity B_i.

$\vartheta_o/\vartheta_{io}$ is plotted against $K \lambda l$ in Figure 6 (Table II).

TABLE II (Fig. 6)

Values of:

$$\frac{\vartheta_o}{\vartheta_{io}} = 1 + \frac{2}{(K\lambda l)^2} - \frac{2 \tanh K\lambda l}{K\lambda l} - \frac{2}{(K\lambda l)^2 \cosh K\lambda l}$$

$K\lambda l$	$\vartheta_o/\vartheta_{io}$	$K\lambda l$	$\vartheta_o/\vartheta_{io}$	$K\lambda l$	$\vartheta_o/\vartheta_{io}$
0.0	0.000	0.8	0.132	8.0	0.779
0.1	0.002	0.9	0.153	10.0	0.820
0.2	0.010	1.0	0.181	12.0	0.847
0.3	0.023	2.0	0.407	16.0	0.882
0.4	0.040	3.0	0.539	20.0	0.905
0.5	0.052	4.0	0.620	25.0	0.923
0.6	0.080	5.0	0.630	30.0	0.936
0.7	0.105	6.0	0.722	–	–

b) Torsional rigidity of the wing.- According to the previous definition, the torsional rigidity of the wing is $B_f = \frac{p b l^2}{2\ \vartheta_o}$. For the wing tip $(x = 0)$ we obtain from equation (20),

$$\vartheta_o = \frac{p\ b\ l^2}{K^2\ (B_1 + B_2)} \left\{ \frac{1}{K^2\ \lambda^2\ l^2} - \frac{\tanh K\ \lambda\ l}{K\ \lambda\ l} + \right.$$

$$\left. + \frac{1}{2} - \frac{1}{K^2\ \lambda^2\ l^2 \cosh K\ \lambda\ l} \right\}.$$

Consequently

$$B_f = \frac{K^2\ (B_1 + B_2)}{1 + \frac{2}{(K\ \lambda\ l)^2} - \frac{2 \tanh K\ \lambda\ l}{K\ \lambda\ l} - \frac{2}{(K\ \lambda\ l)^2 \cosh K\ \lambda\ l}}.$$

As $K\ \lambda\ l$ approaches 0, B_f approaches ∞ and as $K\ \lambda\ l$ approaches ∞, B_f approaches B_i. For ribs without torsional rigidity, the torsional rigidity of the wing therefore coincides with the sum of the torsional rigidities of the spars.

However, if the ribs had absolute torsional rigidity, we would have $B_f = \infty$ for an infinitely long wing, i.e., the wing would not be distorted. B_f/B_i is plotted against $K \lambda l$ in Figure 7 (Table III).

TABLE III (Fig. 7)

Values of:

$$\frac{B_f}{B_i} = \frac{1}{1 + \frac{2}{(K\lambda l)^2} - \frac{2 \tanh K\lambda l}{K\lambda l} - \frac{2}{(K\lambda l)^2 \cosh K\lambda l}}$$

$K\lambda l$	$\frac{B_f}{B_i}$	$K\lambda l$	$\frac{B_f}{B_i}$	$K\lambda l$	$\frac{B_f}{B_i}$
0.0	∞	1.0	5.46	4.5	1.56
0.1	400,00	1.2	4.52	5.0	1.46
0.2	100.00	1.4	3.34	5.5	1.22
0.3	47.60	1.6	3.08	6.0	1.38
0.4	25.00	1.8	2.70	8.0	1.28
0.5	19.20	2.0	2.46	10.0	1.22
0.6	12.50	2.5	2.08	12.0	1.18
0.7	9.54	3.0	1.86	16.0	1.12
0.8	7.68	3.5	1.70	20.0	1.10
0.9	6.54	4.0	1.60	30.0	1.06

c) Torsional rigidity of the wing sections.- This is $B = -\frac{M_t}{\vartheta'} = \frac{p\,b\,x}{\vartheta'}$ (see equation (2)). ϑ' is obtained from equation (21) by differentiation according to x.

$$\vartheta' = \frac{p}{B_i\, K\, \lambda \cosh K\, \lambda\, l} \Big\{K\, \lambda\, (l \cosh K\, \lambda\, x - x \cosh K\, \lambda\, l) - \sinh K\, \lambda\, (l - x)\Big\} \tag{22}$$

Herewith we obtain

$$\frac{B}{B_i} = \frac{K\, \lambda\, x \cosh K\, \lambda\, l}{K\, \lambda\, (l \cosh K\, \lambda\, x - x \cosh K\, \lambda\, l) - \sinh K\, \lambda\, (l - x)}.$$

In Figure 8 (Table IV) B/B_i is plotted against K/l with $K \lambda l$ as parameter. $B/B_i = \infty$ when $x = l$, due to the assumed rigidity of fixation of the wing root.

TABLE IV (Fig. 8)

Values of:

$$\frac{B}{B_i} = \frac{K \lambda x \cosh K \lambda l}{K \lambda (l \cosh K \lambda x - x \cosh K \lambda l) - \sinh K \lambda (l - x)}$$

$\frac{x}{l}$	$K\lambda l=1$	$K\lambda l=2$	$K\lambda l=3$
0.0	0.000	0.000	0.000
0.2	1.748	0.876	0.762
0.4	2.570	1.632	1.240
0.6	5.620	2.536	1.752
0.8	12.488	4.710	2.980
0.9	42.200	9.000	4.320
1.0	∞	∞	∞

d) Course of the torsional moments along the spars.- The formulas $B_1 \vartheta'$ and $B_2 \vartheta'$ give the torsional moments transmitted by a single spar section. Table V gives us the values of ϑ', that is, the angle of torsion per unit length, with respect to the mean angle of torsion $\vartheta_i' = - p\, b\, l/B_i$. We have

$$\frac{\vartheta'}{\vartheta_i'} = \frac{x}{l} - \frac{K \lambda l \cosh K \lambda x - \sinh K \lambda (l - x)}{K \lambda l \cosh K \lambda l}$$

In Figure 9 ϑ/ϑ_i' is plotted against x/l for different values of $K \lambda l$ (Table V).

TABLE V (Fig. 9)

Values of:

$$\frac{\vartheta'}{\vartheta_l'} = \frac{x}{l} - \frac{K\lambda l \cosh K\lambda x - \sinh K\lambda(l - x)}{K\lambda l \cosh K\lambda l}$$

$\frac{x}{l}$	$K\lambda l=1$	$K\lambda l=2$	$K\lambda l=3$	$K\lambda l=4$
0.0	0.113	0.216	0.232	0.250
0.2	0.114	0.228	0.281	0.280
0.4	0.113	0.245	0.317	0.355
0.6	0.097	0.236	0.331	0.418
0.8	0.064	0.170	0.270	0.358
1.0	0.000	0.000	0.000	0.000

The table and curves can be used for plotting the torsional stress of the spars. The maximum torsional stress does not come at the ends of the spars.

e) Course of the shearing forces in the spar sections.– In this case equation (19) reads

$$\bar{Q} = \frac{B_1 + B_2}{b} \vartheta' + p\,x$$

or

$$\frac{\bar{Q}}{p\,x} = 1 + \frac{B_1 + B_2}{b} \vartheta'.$$

Introducing ϑ' from equation (22) we obtain

$$\frac{\bar{Q}}{p\,x} = 1 + \frac{1}{K^2}\left\{\frac{l}{x}\,\frac{\cosh K\lambda x}{\cosh K\lambda l} - \frac{\sinh K\lambda(l - x)}{K\lambda x \cosh K\lambda l} - 1\right\} \qquad (23)$$

For every spar section, the ratio Q/px determines the portion of the external shearing forces held in equilibrium by the tangential stresses in the cross section. For $x = l$, $Q/pl = 1$, that is, the whole external load, which is applied to the spar, is balanced by the tangential stresses in the root section. This indicates that

$$Q_{x=o} + \int_o^l q \, d \, x = 0,$$

i.e., the resultant of the forces acting on the spars, as the result of their being connected by the ribs, is zero.* In other words, in a rigidly mounted wing the spars cannot mutually relieve one another. Q/px is plotted against x/l in Figure 10, for different values of $K \lambda l$ and K (Table VI).

TABLE VI (Fig. 10)

Values of:

$$\frac{\bar{Q}}{p\,x} = 1 + \frac{1}{K^2}\left(\frac{l}{x}\;\frac{\cosh K\lambda x}{\cosh K\lambda l} - \frac{\sinh K\lambda (l - x)}{K\lambda x \cosh K\lambda l} - 1\right)$$

$\frac{x}{l}$	$K \lambda l = 1$			$K \lambda l = 2$		
	K=1	K=1.2	K=1.4	K=1	K=1.2	K=1.4
0.0	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
0.1	-0.15	0.202	0.416	-6.83	-4.44	-3.00
0.2	0.45	0.618	0.720	-2.41	-1.37	-0.74
0.3	0.62	0.739	0.808	0.21	0.44	0.60
0.4	0.72	0.806	0.857	0.39	0.57	0.69
0.5	0.78	0.825	0.890	0.51	0.65	0.75
0.6	0.83	0.882	0.914	0.50	0.72	0.79
0.8	0.92	0.946	0.960	0.79	0.85	0.89
1.0	1.00	1.000	1.000	1.00	1.00	1.00

f) <u>Course of the load reaction q along the spars</u>.- For every spar section at the distance x from the wing tip, we have the equilibrium condition

$$p\,x - Q = Q_{x=o} + \int_o^x q \, d \, x$$

*First discovered by Biezeno, Koch and Koning ("Ueber die Berechnung von freitragenden Flugzeugflügeln." Zeitsch. f. angew. Math. u. Mech., 1926, pp. 97-105.)

According to equation (22)

$$1 - \frac{Q}{p\,x} = -\frac{1}{K^2}\left\{\frac{l}{x}\,\frac{\cosh K\lambda x}{\cosh K\lambda l} - \frac{\sinh K\lambda(l - x)}{K\lambda x\cosh K\lambda l} - 1\right\}$$

and hence

$$Q_{x=0} + \int_0^x q\,d\,x = -\frac{q}{K^2}\left\{l\,\frac{\cosh K\lambda x}{\cosh K\lambda l} - \frac{\sinh K\lambda(l - x)}{K\lambda\cosh K\lambda l} - x\right\};$$

By derivation according to x

$$q = -\frac{p}{K^2}\left\{K\lambda l\,\frac{\sinh K\lambda x}{\cosh K\lambda l} + \frac{\cosh K\lambda(l - x)}{\cosh K\lambda l} - 1\right\}.$$

For $x = 0$ (wing tip), $q = 0$; For $x = l$ (wing root),

$$q = -\frac{p}{K^2}\left\{K\lambda l\tanh K\lambda l + \frac{1}{\cosh K\lambda l} - 1\right\}.$$

Figure 11 shows the curve of

$$K^2\,\frac{q}{p} = 1 - \frac{K\lambda l\sinh K\lambda x + \cosh K\lambda(l - x)}{\cosh K\lambda l}$$

plotted against x/l for different values of $K\lambda l$. See Table VII for the computed values. As shown by Figure 11, q indicates an increase in the load p toward the wing tip and a decrease toward the wing root.

TABLE VII (Fig. 11)

Values of:

$$K^2 \frac{q}{p} = 1 - \frac{K \lambda l \sinh K \lambda x + \cosh K \lambda (l - x)}{\cosh K \lambda l}$$

$\frac{x}{l}$	$K\lambda l=1$	$K\lambda l=2$	$K\lambda l=3$
0.0	0.000	0.000	0.000
0.2	0.010	0.090	0.259
0.4	-0.034	0.047	0.242
0.6	-0.112	-0.157	-0.055
0.8	-0.237	-0.547	-0.745
1.0	-0.410	-1.191	-2.099

g) Dependence of force Q_0 on $K \lambda l$.- For $x = 0$, equation (19) gives

$$Q_0 = \frac{B_1 + B_2}{b} \vartheta_0'$$

Introducing $x = 0$ into equation (22) gives

$$\vartheta_0' = \frac{p b l}{K^2 (B_1 + B_2) \cosh K \lambda l} \left\{1 - \frac{\sinh K \lambda l}{K \lambda l}\right\},$$

whence we obtain

$$\frac{Q_0}{p l} = \frac{1}{K^2} \left(\frac{1}{\cosh K \lambda l} - \frac{\tanh K \lambda l}{K \lambda l}\right).$$

As $K \lambda l$ approaches ∞, $Q_0/p l$ approaches 0. $Q_0/p l$ is plotted against $K \lambda l$ in Figure 12. The curve reaches its maximum value at about $K \lambda l = 3$.

TABLE VIII (Fig. 12)

Values of:

$$\frac{Q_0}{p l} = \frac{1}{K^2}\left(\frac{1}{\cosh K \lambda l} - \frac{\tanh K \lambda l}{K \lambda l}\right)$$

$K\lambda l$	K=1	K=1.2	K=1.4
0	0.000	0.000	0.000
1	-0.113	-0.077	-0.057
2	-0.216	-0.150	-0.110
3	-0.233	-0.165	-0.118
4	-0.213	-0.148	-0.108
5	-0.186	-0.129	-0.095
6	-0.162	-0.112	-0.082
8	-0.124	-0.086	-0.063
10	-0.100	-0.069	-0.051

Case II. Wing Subjected to a Constant Torsional Moment Applied to Its Tip

We shall assume that the wing is subjected to the action of a torsional moment, exerted, e.g., by two equal and opposite forces acting on the spars. These produce the same torsional moment $M_t = P b$, in every wing section. Since $p_1 = p_2 = 0$, the fundamental equation (12) now becomes

$$\vartheta'''' - K^2 \lambda^2 \vartheta'' = 0.$$

The general integral of this equation is

$$\vartheta = C_1 e^{K\lambda \bar{x}} + C_2 e^{-K\lambda \bar{x}} + C_3 \bar{x} + C_4,$$

where C_1, C_2, C_3 and C_4 are the four integration constants. The limiting conditions for this case are

$$\vartheta_{x=l} = 0$$

$$\vartheta'_{x=l} = 0$$

$$\vartheta''_{x=o} = 0$$

$$\vartheta'''_{x=o} - K^2 \lambda^2 \vartheta'_{x=o} = \frac{M_t}{A b^2}$$

whence we obtain

$$C_1 = \frac{M_t}{2 A b^2 K^3 \lambda^3 \cosh K \lambda l}$$

$$C_2 = - \frac{M_t}{2 A b^2 K^3 \lambda^3 \cosh K \lambda l}$$

$$C_3 = - \frac{M_t}{A b^2 K^2 \lambda^2}$$

$$C_4 = - \frac{M_t l}{K^2 \lambda^2 A b^2} \left(\frac{\tanh K \lambda l}{K \lambda l} - 1 \right)$$

and

$$\vartheta = \frac{M_t}{K^3 \lambda^3 A b^2 \cosh K \lambda l} \sinh K \lambda x - \sinh K \lambda l + $$

$$+ K \lambda (l - x) \cosh K \lambda l \qquad (24)$$

<u>a) Course of the torsional angle along the wing.</u>- If we introduce $\lambda^2 = \frac{B_1 + B_2}{A b^2}$ as in equation (12), we can then write equation (24) in the form

$$\vartheta = \frac{M_t l}{K^2 (B_1 + B_2) K \lambda l \cosh K \lambda l} \left\{ \sinh K \lambda x - \right.$$

$$\left. - \sinh K \lambda l + K \lambda l (l - x) \cosh K \lambda l \right\} \qquad (24a)$$

According to St. Venant's theory

$$\vartheta_{io} = \frac{M_t \, l}{K^2 \, (B_1 + B_2)}$$

indicates the maximum angle of torsion of a girder of length l and torsional rigidity $B_i = K^2 \, (B_1 + B_2)$. Then

$$\frac{\vartheta}{\vartheta_{io}} = \frac{1}{K \lambda \, l \cosh K \lambda \, l} \left\{ \sinh K \lambda \, x - \right.$$

$$\left. - \sinh K \lambda \, l + K \lambda \, (l - x) \cosh K \lambda \, l \right\} \qquad (25)$$

In Figure 13, ϑ/ϑ_{io} is plotted against x/l for various constant values of $K \lambda \, l$. The dash-dot line here represents the function $\vartheta_i/\vartheta_{io}$, in which ϑ_i is the angle of torsion along the girder.

TABLE IX (Fig. 13)

Values of:

$$\frac{\vartheta}{\vartheta_{io}} = 1 - \frac{x}{l} - \frac{\tanh K \lambda \, l}{K \lambda \, l} + \frac{\sinh K \lambda \, x}{K \lambda \, l \cosh K \lambda \, l}$$

$\frac{x}{l}$	$K\lambda l=1$	$K\lambda l=2$	$K\lambda l=3$
0.0	0.240	0.518	0.672
0.2	0.170	0.373	0.490
0.4	0.105	0.236	0.318
0.5	0.050	0.118	0.167
0.8	0.014	0.032	0.049
1.0	0.000	0.000	0.000

From equation (25) we obtain for $x = 0$

$$\frac{\vartheta_o}{\vartheta_{io}} = 1 - \frac{\tanh K \lambda \, l}{K \lambda \, l} .$$

$\vartheta_o/\vartheta_{io}$ is plotted against $K \lambda l$ in Figure 14 (Table X). This ratio approaches 1, as $K \lambda l$ approaches ∞. In this case the wing behaves like an ordinary girder with a torsional rigidity equal to the sum of the torsional rigidities of the spars and ribs.

TABLE X (Fig. 14)

Values of:

$$\frac{\vartheta_o}{\vartheta_{io}} = 1 - \frac{\tanh K \lambda l}{K \lambda l}$$

$K\lambda l$	$\frac{\vartheta_o}{\vartheta_{io}}$	$K\lambda l$	$\frac{\vartheta_o}{\vartheta_{io}}$	$K\lambda l$	$\frac{\vartheta_o}{\vartheta_{io}}$
0.0	0.000	1.0	0.240	4.5	0.779
0.1	0.003	1.2	0.306	5.0	0.800
0.2	0.014	1.4	0.367	5.5	0.818
0.3	0.024	1.6	0.426	6.0	0.835
0.4	0.050	1.8	0.473	-	-
0.5	0.075	2.0	0.515	-	-
0.6	0.105	2.5	0.604	-	-
0.7	0.136	3.0	0.679	-	-
0.8	0.170	3.5	0.714	-	-
0.9	0.205	4.0	0.749	-	-

b) Torsional rigidity of the wing.- This is $B_f = \frac{M_t l}{\vartheta_o}$.

For $x = 0$, equation (24a) gives

$$\vartheta_o = \frac{M_t l}{K^2 (B_1 + B_2)} \left\{1 - \frac{\tanh K \lambda l}{K \lambda l}\right\}.$$

Hence

$$B_f = \frac{B_i}{1 - \frac{\tanh K \lambda l}{K \lambda l}}.$$

B_f/B_i is plotted against $K \lambda l$ in Figure 15 (Table XI).

B_f approaches ∞, as $K \lambda l$ approaches 0.

B_f approaches B_i, as $K \lambda l$ approaches ∞.

TABLE XI (Fig. 15)

Values of:

$$\frac{B_f}{B_i} = \frac{1}{1 - \frac{\tanh K\lambda l}{K\lambda l}}$$

$K\lambda l$	$\frac{B_f}{B_i}$	$K\lambda l$	$\frac{B_f}{B_i}$	$K\lambda l$	$\frac{B_f}{B_i}$
0.0	∞	1.0	4.18	4.5	1.28
0.1	303.00	1.2	3.27	5.0	1.25
0.2	71.50	1.4	2.72	5.5	1.22
0.3	35.70	1.6	2.35	5.0	1.20
0.4	20.00	1.8	2.11	-	-
0.5	13.20	2.0	1.94	-	-
0.6	9.52	2.5	1.65	-	-
0.7	7.36	3.0	1.49	-	-
0.8	5.90	3.5	1.40	-	-
0.9	4.88	4.0	1.33	-	-

c) Torsional rigidity of the wing section.- This is $B = -M_t/\vartheta'$ (cf. equation (2)). When differentiated with respect to x (equation (24a)),

$$\vartheta' = \frac{M_t}{K^2 (B_1 + B_2) \cosh K\lambda l} \quad \cosh K\lambda x - \cosh K\lambda l\} \qquad (26)$$

Hence

$$B = \frac{K^2 (B_1 + B_2)}{1 - \frac{\cosh K\lambda x}{\cosh K\lambda l}},$$

and

$$\frac{B}{K^2 (B_1 + B_2)} = \frac{B}{B_i} = \frac{1}{1 - \frac{\cosh K\lambda x}{\cosh K\lambda l}}.$$

At the wing root, due to the assumed rigid fixation, we have $B/B_i = \infty$. B/B_i is plotted against x/l for different values of $K\lambda l$ in Figure 16 (Table XII).

TABLE XII (Fig. 16)

Values of:

$$\frac{B}{B_i} = \frac{1}{1 - \dfrac{\cosh K \lambda x}{\cosh K \lambda l}}$$

$\frac{x}{l}$	$K\lambda l=1$	$K\lambda l=2$	$K\lambda l=3$
0.0	2.840	1.360	1.115
0.2	2.950	1.400	1.142
0.4	3.350	1.555	1.220
0.6	4.320	1.920	1.445
0.8	7.500	3.18	2.230
0.9	14.300	5.75	3.780
1.0	∞	∞	∞

d) Course of the shearing forces in the spar sections along the wing.- In this case equation (19) reads

$$Q = \frac{B_1 + B_2}{b} \vartheta'' + P.$$

Using equation (26) for ϑ' we obtain

$$Q = \frac{P}{K^2 \cosh K \lambda l} \left\{ \cosh K \lambda x - \cosh K \lambda l \right\} + P$$

and

$$\frac{Q}{P} = 1 - \frac{1}{K^2} \left(1 - \frac{\cosh K \lambda x}{\cosh K \lambda l}\right) \tag{27}$$

For $x = l$ (wing root) we obtain $Q/P = 1$. For $x = 0$ (wing tip) we obtain

$$\frac{Q_0}{P} = 1 - \frac{1}{K^2} \left(1 - \frac{1}{\cosh K \lambda l}\right).$$

For very long or very flexible wings this equation can be written

$$\frac{Q}{P} = 1 - \frac{1}{K^2}$$

If $K = 1$, that is, if the torsional rigidity of the ribs is 0, we obtain $Q/P = 0$ for the wing tip, i.e., $Q = 0$. This means that the torsional moment of the wing near its tip is balanced only by the torsion of the spars.

In the case when the ribs have an infinite torsional rigidity, $Q = P$ for every spar section. The torsional moment is then balanced only by the shearing forces acting in the spar sections. The shearing force Q_o at the wing tip for $x = 0$ (equation (27)) is

$$Q_o = P \left[1 - \frac{1}{K^2} \left(1 - \frac{1}{\cosh K \lambda l}\right)\right].$$

Q_o approaches $P \left(1 - \frac{1}{K^2}\right)$, as $K \lambda l$ approaches ∞.

Q_o approaches P, as K approaches ∞.

Q_o/P is plotted against $K \lambda l$ in Figure 17 (Table XIII).

TABLE XIII (Fig. 17)

Values of:

$$\frac{Q_o}{P} = 1 - \frac{1}{K^2} \left(1 - \frac{1}{\cosh K \lambda l}\right)$$

$K\lambda l$	K=1	K=1.2	K=1.4
0	1.000	1.000	1.000
1	0.648	0.756	0.821
2	0.266	0.490	0.626
3	0.099	0.375	0.540
4	0.036	0.330	0.510
5	0.013	0.314	0.496
6	0.009	0.310	0.493
8	~ 0.000	0.307	0.491
10	~ 0.000	0.306	0.490

From these considerations it follows that an increase in the torsional rigidity of the ribs entails an increase in the torsional rigidity of the wing and consequently a diminution of the torsional moment accompanied by a simultaneous increase of the shearing forces in the spars. Q/P is plotted against x/l at different values of $K \lambda l$ and K in Figure 18 (Table XIV).

TABLE XIV (Fig. 18)

Values of:

$$\frac{Q}{P} = 1 - \frac{1}{K^2}\left(1 - \frac{\cosh K \lambda x}{\cosh K \lambda l}\right)$$

$\frac{x}{l}$	$K \lambda l = 1$			$K \lambda l = 2$			$K \lambda l = 3$		
	K=1	K=1.2	K=1.4	K=1	K=1.2	K=1.4	K=1	K=1.2	K=1.4
0.0	0.648	0.756	0.821	0.266	0.490	0.626	0.099	0.375	0.520
0.2	0.661	0.765	0.827	0.288	0.505	0.637	0.117	0.387	0.550
0.4	0.700	0.792	0.847	0.358	0.555	0.671	0.180	0.230	0.581
0.6	0.768	0.839	0.881	0.487	0.648	0.735	0.307	0.519	0.646
0.8	0.868	0.908	0.932	0.686	0.782	0.840	0.550	0.687	0.770
1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

e) Course of the supporting reaction q along the wing.– It was found that equation (27) which, after transformation, reads

$$Q = P\left[1 - \frac{1}{K^2}\left(1 - \frac{\cosh K \lambda x}{\cosh K \lambda l}\right)\right],$$

holds good for every spar section. Q is that portion of P which is balanced in every spar section by the tangential stresses The other part of P, namely, $P \frac{1}{K^2}\left(1 - \frac{\cosh K \lambda x}{\cosh K \lambda l}\right)$ is balanced by the forces transmitted by the ribs to the portion of the spar between the wing tip and the given cross section. Therefore we may write

$$Q_0 + \int_0^x q\, d\, x = - P \frac{1}{K^2}\left(1 - \frac{\cosh K \lambda x}{\cosh K \lambda l}\right)$$

by differentiating which we obtain

$$q = \frac{M_t}{b} \frac{\lambda}{K^2} \frac{\sinh K \lambda x}{\cosh K \lambda l} .$$

$q = 0$ for $x = 0$ (wing tip).

$q = \frac{M_t}{b} \frac{\lambda}{K^2} \tanh K \lambda l$ for $x = l$ (wing root) .

q approaches 0 as K approaches ∞, that is, for ribs with absolute torsional rigidity. $q \frac{b K^2}{\lambda M_t}$ is plotted against x/l for different values of $K \lambda l$ in Figure 19 (Table XV).

TABLE XV (Fig. 19)

Values of:

$$q \frac{b K^2}{\lambda M_t} = \frac{\sinh K \lambda x}{\cosh K \lambda l}$$

$\frac{x}{l}$	$K\lambda l=1$	$K\lambda l=2$	$K\lambda l=3$
0.0	0.000	0.000	0.000
0.2	0.130	0.109	0.063
0.4	0.266	0.236	0.150
0.6	0.412	0.402	0.292
0.8	0.576	0.632	0.542
1.0	0.762	0.964	0.996

f) Course of the torsional moment along the spars.- Equation (26) was transformed into

$$\vartheta' \frac{K^2 (B_1 + B_2)}{M_t} = \frac{\cosh K \lambda x}{\cosh K \lambda l} - 1 \qquad (28)$$

According to St. Venant's theory, the ratio

$$\frac{M_t}{K^2 (B_1 + B_2)} = \frac{M_t}{B_i}$$

represents the constant increment of the torsional angle of a cylindrical girder of torsional rigidity B_i. We put

$$\frac{M_t}{B_i} = - \vartheta_i',$$

with which equation (28) can be written

$$\frac{\vartheta'}{\vartheta_i'} = 1 - \frac{\cosh K \lambda x}{\cosh K \lambda l}.$$

TABLE XVI (Fig. 20)

Values of:

$$\frac{\vartheta'}{\vartheta_i'} = 1 - \frac{\cosh K \lambda x}{\cosh K \lambda l}$$

$\frac{x}{l}$	$K\lambda l=1$	$K\lambda l=2$	$K\lambda l=3$	$K\lambda l=4$
0.0	0.352	0.735	0.896	0.964
0.2	0.339	0.714	0.876	0.950
0.4	0.299	0.644	0.820	0.906
0.6	0.232	0.520	0.692	0.796
0.8	0.133	0.315	0.450	0.550
1.0	0.000	0.000	0.000	0.000

This ratio is evidently proportional to the torsional moment in every section. ϑ'/ϑ_i' is plotted against x/l for different values of $K \lambda l$ in Figure 20 (Table XVI).

S u m m a r y

The present paper is confined to the case of a cantilever wing with constant spar and rib sections. Although this assumption is seldom borne out in practice, nevertheless the investigation of this simple case enables us to gain a clearer idea of how the distortion of a two-spar wing is effected.

The quantity

$$\lambda\ l = \frac{l}{b} \sqrt{\frac{B_1 + B_2}{A}}$$

first enters into the computation as the essential parameter. The tables and figures show that, for small values of this proportionality factor, the distortion depends chiefly on the bending rigidity of the spars. In other words, the distortion is the result of the difference in the deflection of the two spars.

At large values of $\lambda\ l$, however, the wing behaves essentially like a twisted girder whose torsional rigidity equals the sum of the torsional rigidities of the spars (strengthened to a certain extent by the torsional rigidity of the ribs). The second parameter

$$K = \sqrt{1 + \frac{\bar{b}}{d} \frac{B_r}{B_1 + B_2}}$$

as previously remarked, expresses the effect of the torsional rigidity of the ribs. The tables furnish accurate information regarding this effect. For large values of $\lambda\ l$, this effect can be so expressed that it is only necessary to add to the tor-

sional rigidity of the spars the torsional rigidity of the ribs per length unit b (distance between the spars).

All the calculations were made on the assumption that the system of ribs can be replaced by a uniformly distributed elastic union. It is a further task for theory and experimentation to determine to what extent this assumption will give sufficiently accurate results for practical purposes.

Translation by Dwight M. Miner,
National Advisory Committee
for Aeronautics.

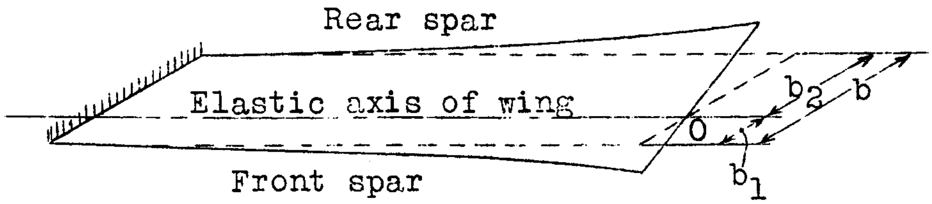


Fig.1

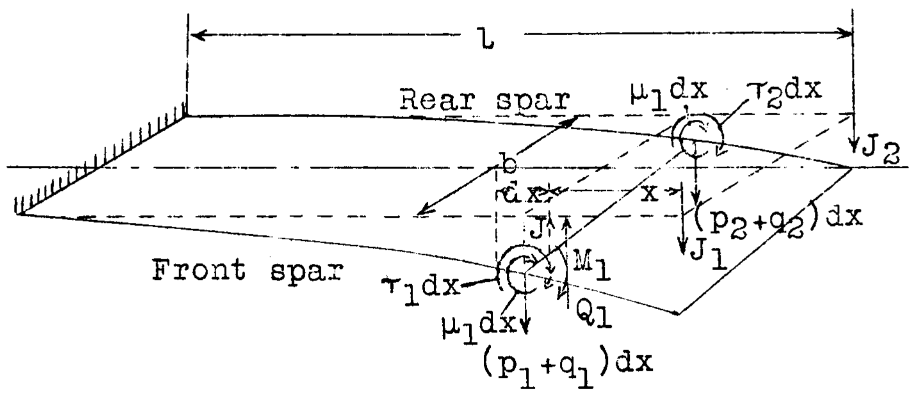


Fig.2

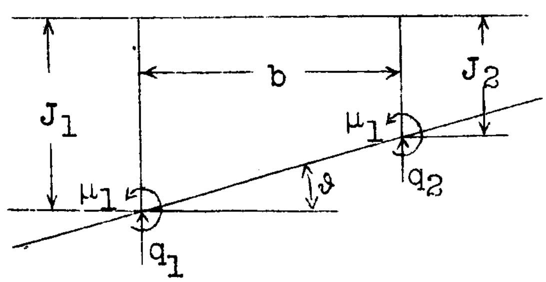


Fig.3

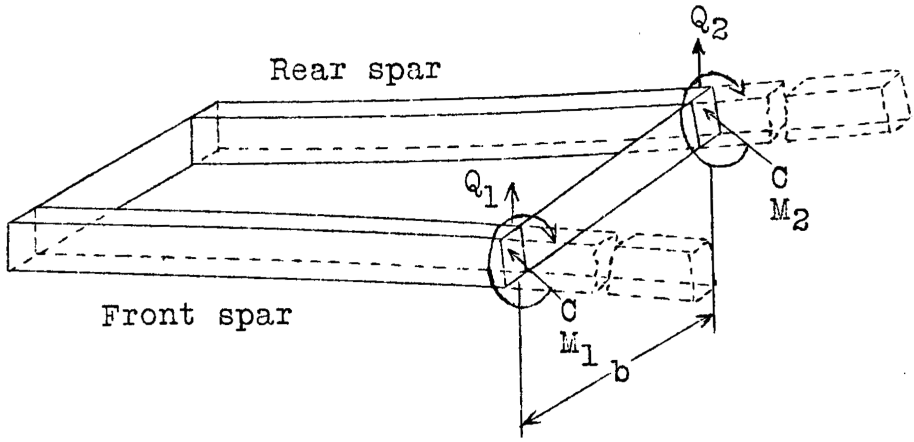

Rear spar
Q_2
Q_1
C
M_2
Front spar
C
M_1
b

Fig.4

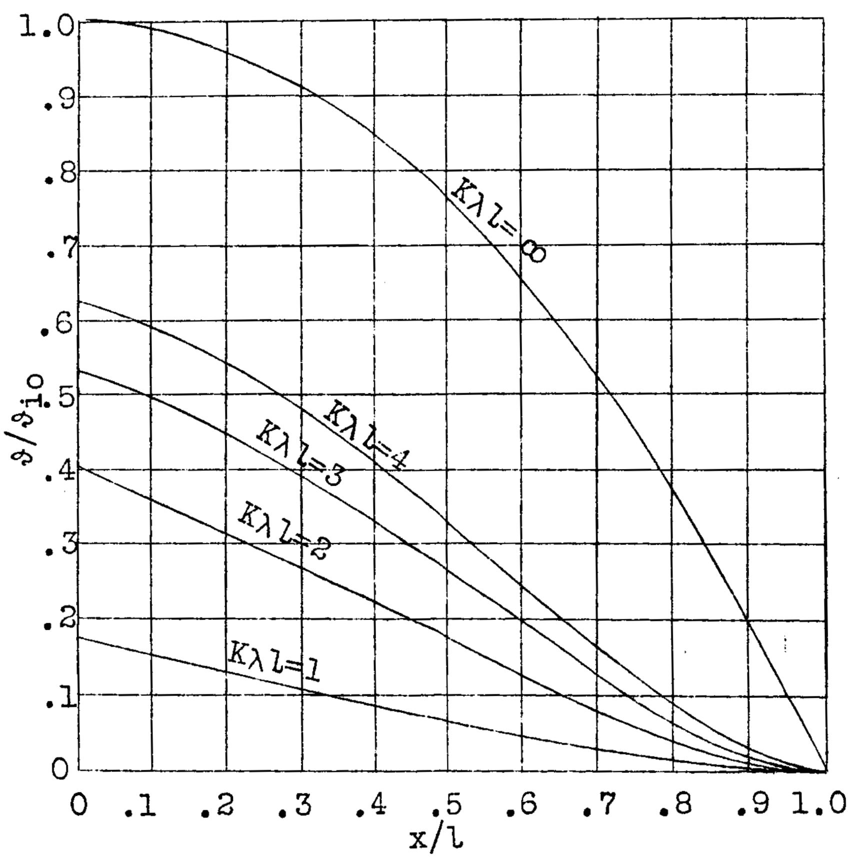

1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
ϑ/ϑ_{10}
$K\lambda l=\infty$
$K\lambda l=4$
$K\lambda l=3$
$K\lambda l=2$
$K\lambda l=1$
0 .1 .2 .3 .4 .5 .6 .7 .8 .9 1.0
x/l

Fig.5

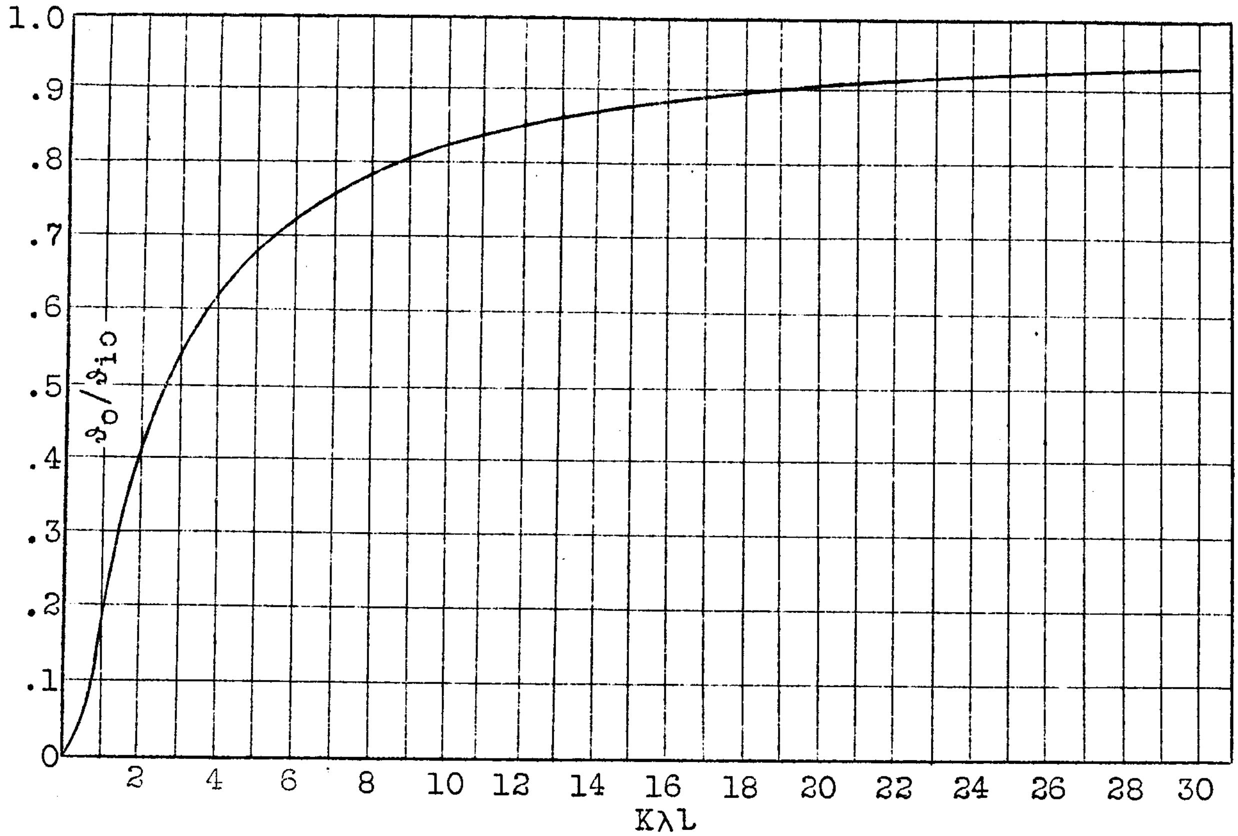

1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
$\vartheta_o/\vartheta_{io}$
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
Kλl

Fig.6

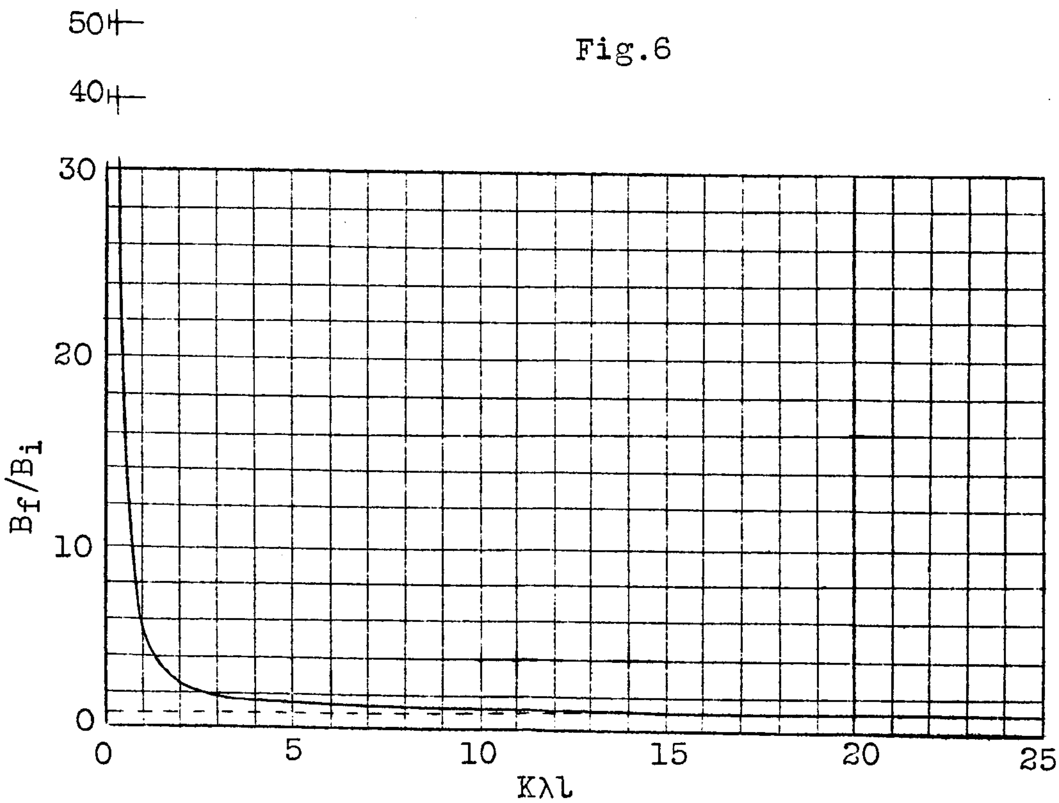

50
40
30
20
10
0
B_f/B_i
0
5
10
15
20
25
Kλl

Fig.7

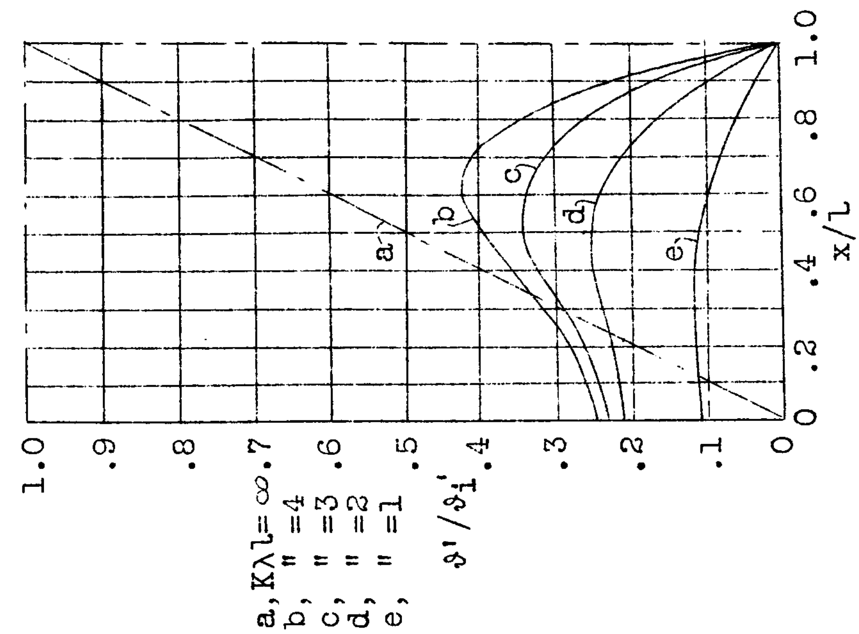


Fig.9

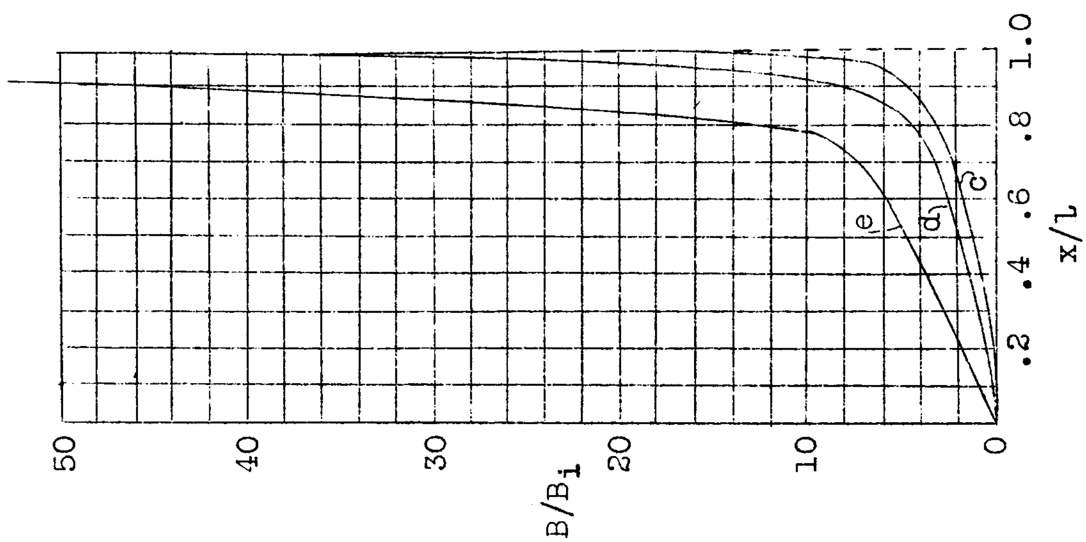


Fig.8

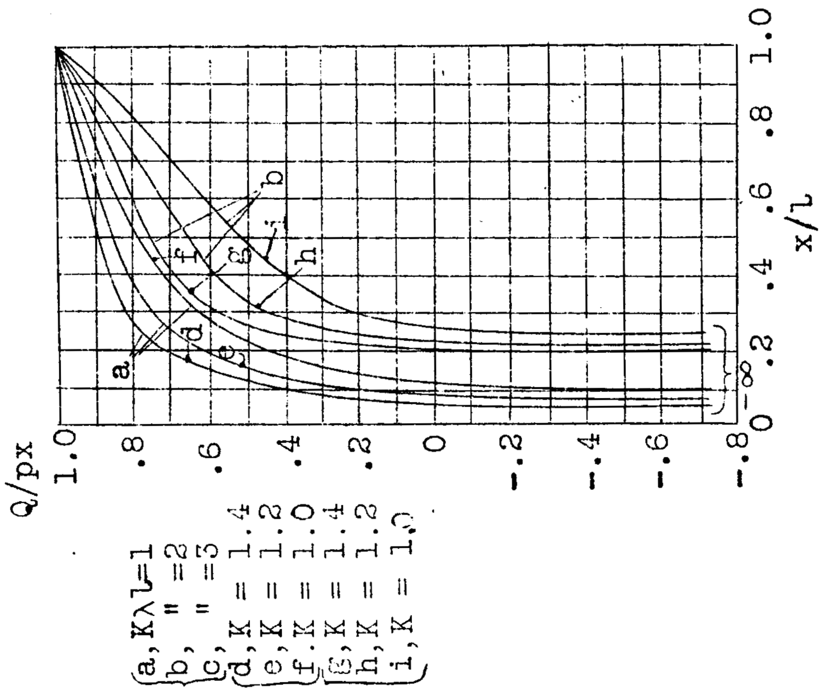


Fig.10

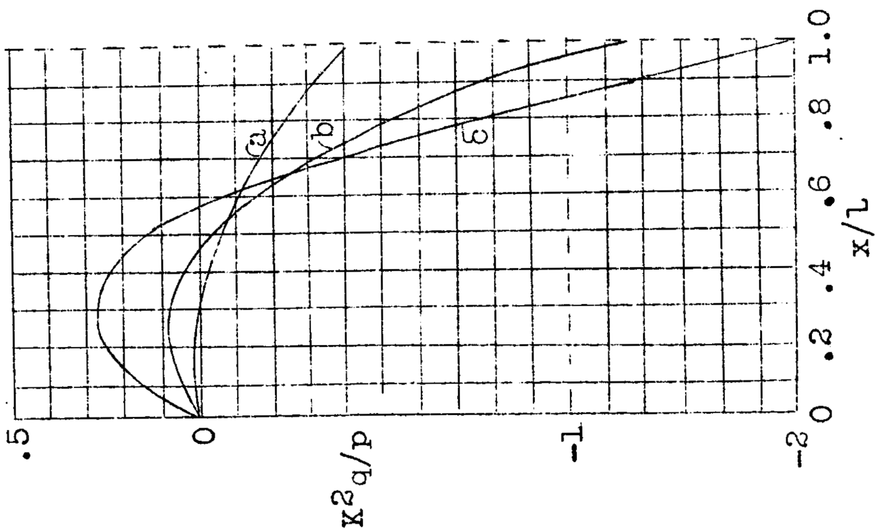


Fig.11

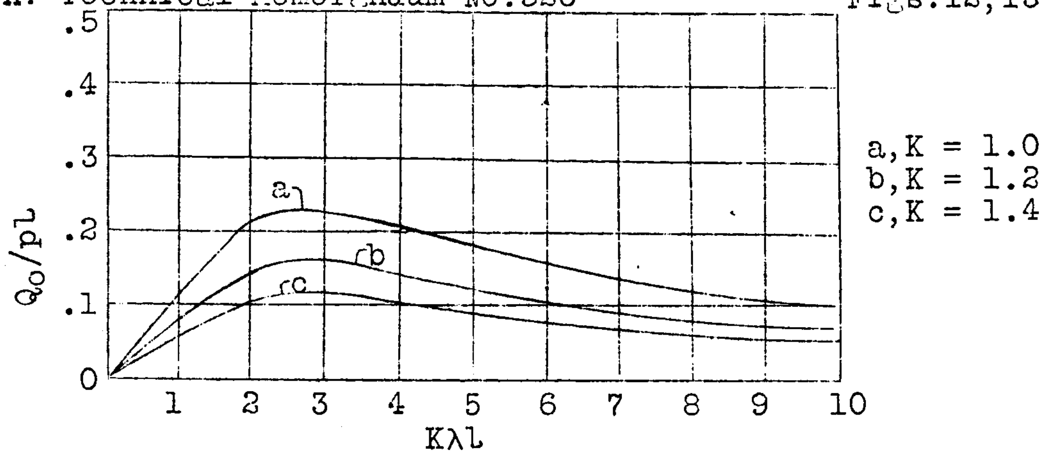


Fig.12

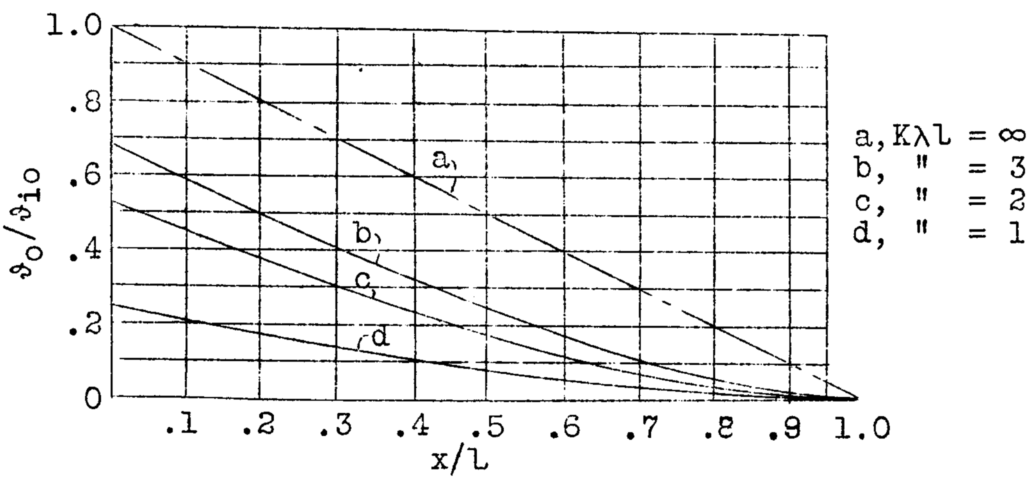


Fig.13

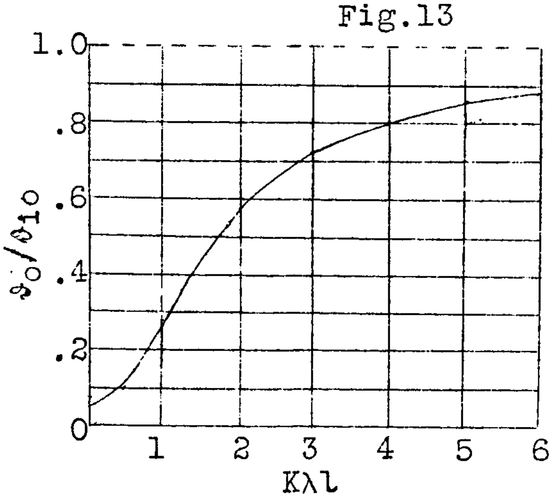
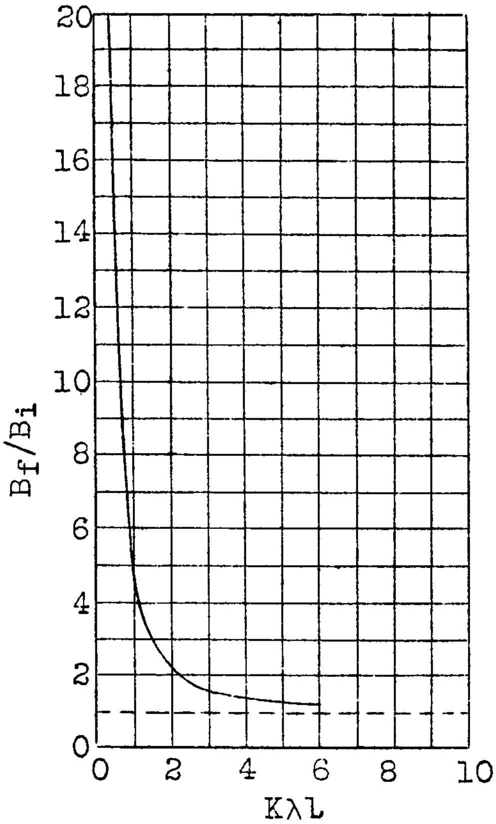
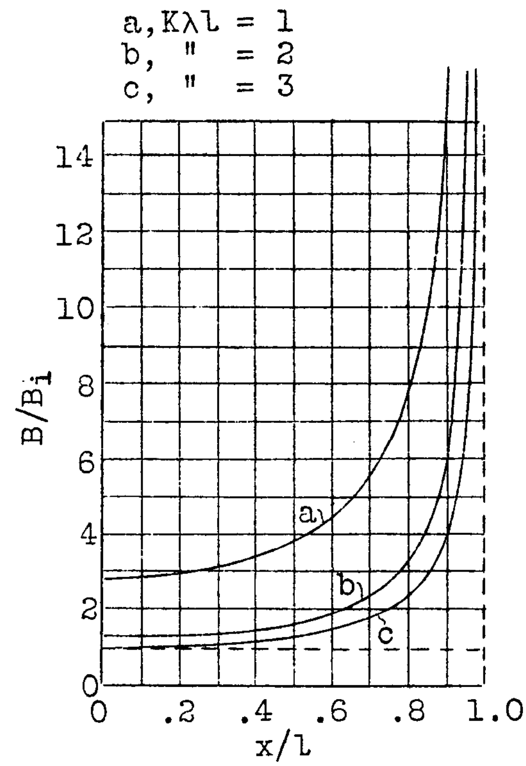


Fig.14



20
18
16
14
12
10
8
6
4
2
0
B_f/B_i
0
2
4
6
8
10
$K\lambda l$

Fig.15



a, $K\lambda l$ = 1
b, " = 2
c, " = 3
14
12
10
8
6
4
2
0
B/B_i
a
b
c
0
.2
.4
.6
.8
1.0
x/l

Fig.16

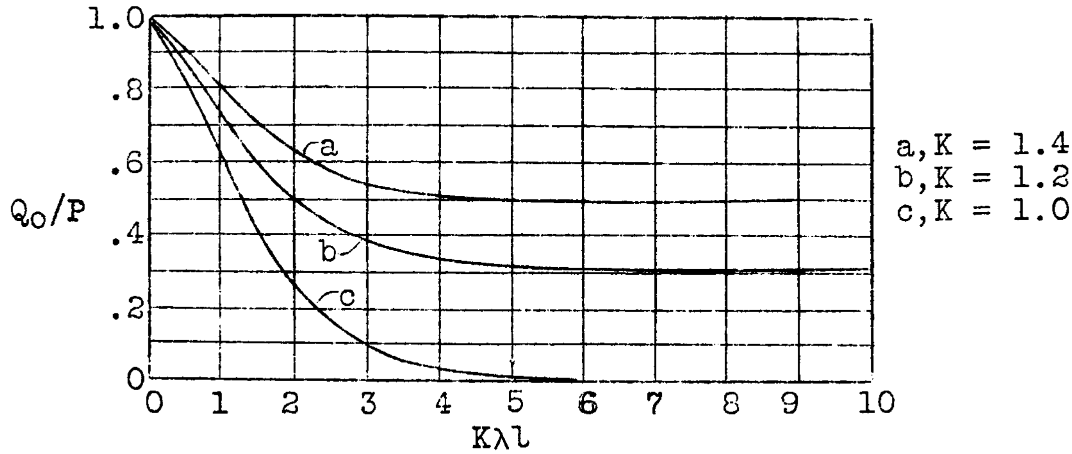

1.0
.8
.6
.4
.2
0
Q_0/P
a
b
c
0 1 2 3 4 5 6 7 8 9 10
$K\lambda l$
a, K = 1.4
b, K = 1.2
c, K = 1.0

Fig.17

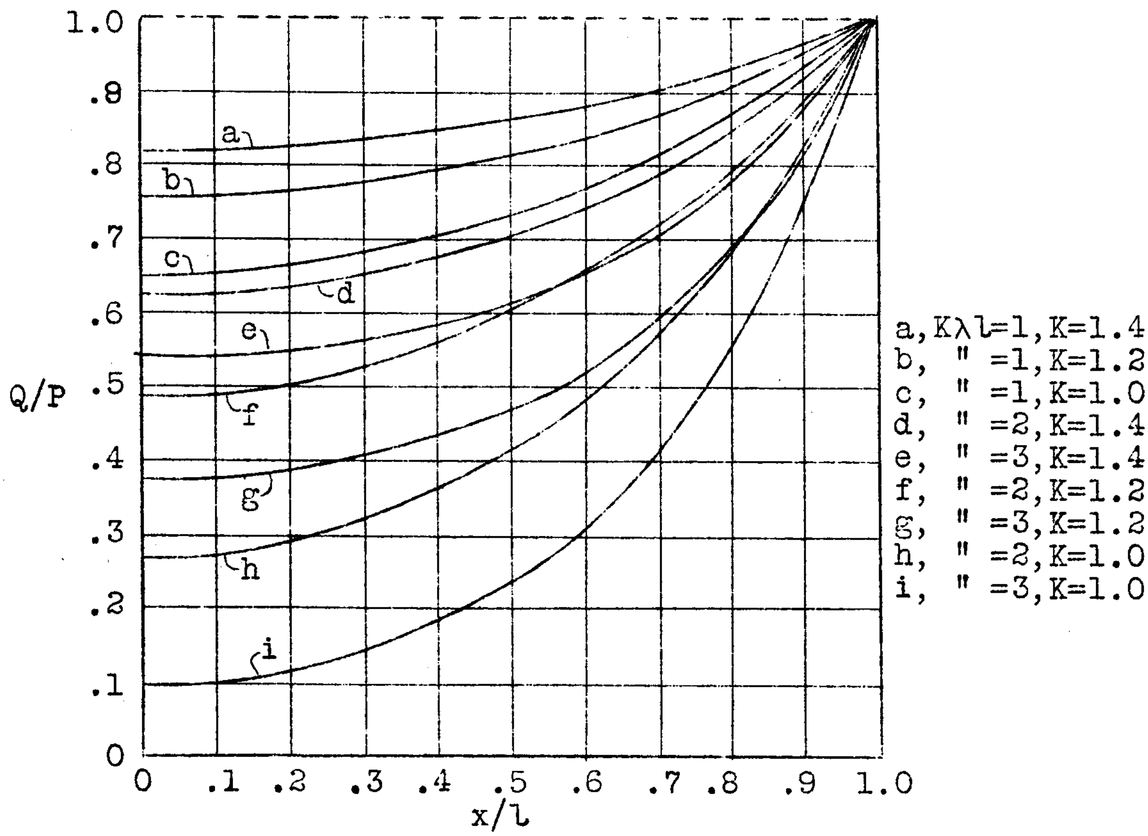
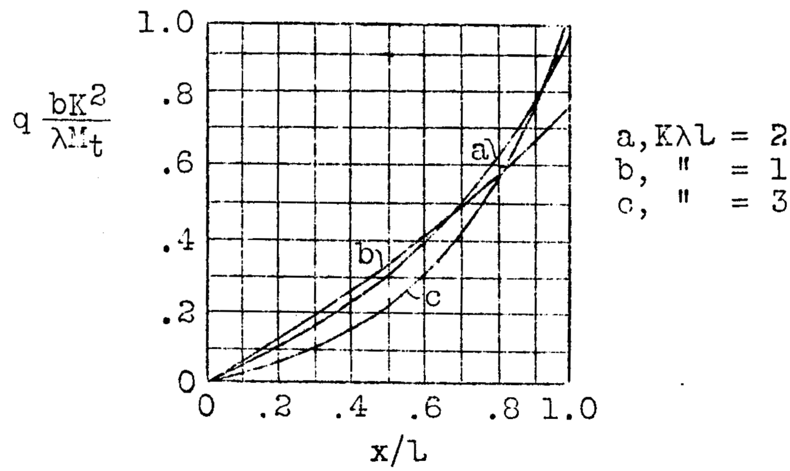
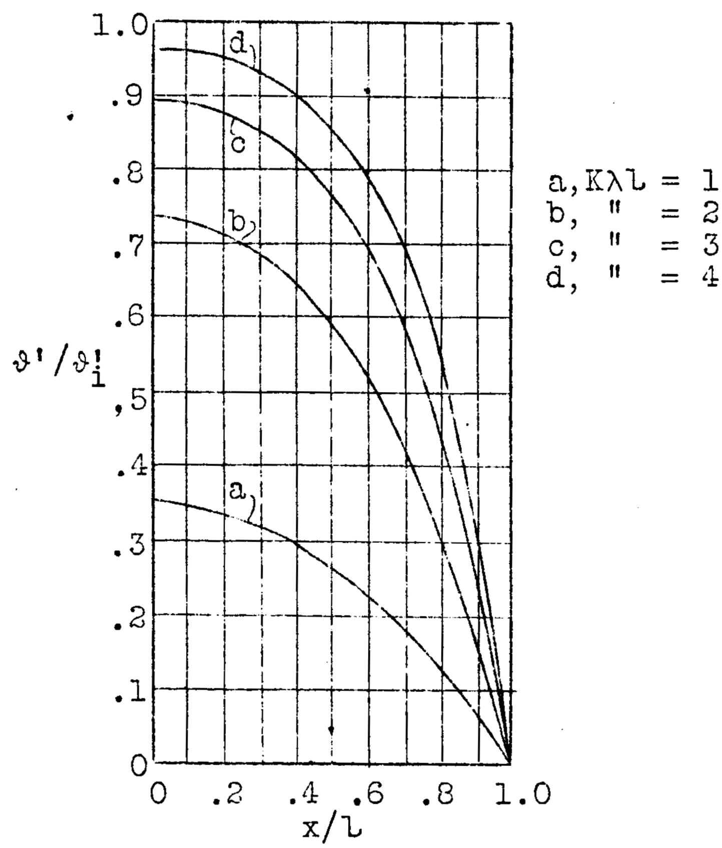

1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
Q/P
a
b
c
d
e
f
g
h
i
0 .1 .2 .3 .4 .5 .6 .7 .8 .9 1.0
x/l
a, $K\lambda l$=1, K=1.4
b, " =1, K=1.2
c, " =1, K=1.0
d, " =2, K=1.4
e, " =3, K=1.4
f, " =2, K=1.2
g, " =3, K=1.2
h, " =2, K=1.0
i, " =3, K=1.0

Fig.18



1.0
.8
.6
.4
.2
0
$q \frac{bK^2}{\lambda M_t}$
a)
b)
c
0 .2 .4 .6 .8 1.0
x/l
a, Kλl = 2
b, " = 1
c, " = 3

Fig.19



1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
ϑ'/ϑ'_i
d)
c
b)
a)
0 .2 .4 .6 .8 1.0
x/l
a, Kλl = 1
b, " = 2
c, " = 3
d, " = 4

Fig.20